J Glob Optim (2008) 40:495–500
DOI 10.1007/s10898-007-9249-7

On the maximum principle for linear parabolic equations

Hung-Ju Kuo · Neil S. Trudinger

Received: 20 September 2007 / Accepted: 21 September 2007 / Published online: 11 October 2007

Abstract We prove extensions of our previous estimates for linear elliptic equations with inhomogeneous terms in L^p spaces, $p \le n$ to linear parabolic equations with inhomogeneous terms in L^p, $p \le n + 1$. As with the elliptic case, our results depend on restrictions on parabolicity determined by certain subcones of the positive cone . They also extend the maximum principle of Krylov for the case $p = n+1$, corresponding to the usual parabolicity.

Keywords Maximum principles · Linear parabolic equations · Parabolic Hessian equation · Hessian integrals

1 Introduction

In our previous paper [8], we considered extensions of the maximum principles of Aleksandrov, Bakel' man and Pucci, (see [2]) for linear elliptic operators to lower L^p exponents under restricted ellipticity hypotheses. In the present paper we consider analogous results for linear parabolic operators, thereby providing corresponding extensions to the Krylov maximum principle. Specifically, we consider linear parabolic operators of the form

$$Pu := D_t u - a^{ij}(x,t) D_{ij} u, \tag{1}$$

in bounded domains $\mathcal{D}$ in space-time $\mathbb{R}^{n-1}(x,t)$, where the matrix function $\mathcal{A} = [a^{ij}] : \mathcal{D} \to \mathcal{S}^n$ is positive in $\mathcal{D}$. Here $D_t u$ and $D^2 u = [D_{ij} u]$ denote respectively the time derivative and the spatial Hessian of an appropriately smooth function $u : \mathcal{D} \to \mathbb{R}$. As in [8],

H.-J. Kuo
Department of Applied Mathematics, National Chung-Hsing University, Taichung 402, Taiwan
e-mail: kuohj@nchu.edu.tw

N. S. Trudinger (✉)
Centre for Mathematics and its Applications, Australian National University, Canberra, ACT 0200, Australia
e-mail: neil.trudinger@anu.edu.au

our restrictions on parabolicity are expressed in terms of the subcones of the positive cone, determined by the elementary symmetric functions $S_k,\ k = 1, \dots, n$, given by

$$S_k(\lambda) = \sum \lambda_{i_1}, \dots, \lambda_{i_k} \tag{2}$$

for $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, where the summation is taken over all increasing k tuples $(i_1, \dots, i_k) \subset \{1, \dots, n\}$. The cone Γ_k is then defined by

$$\Gamma_k = \left\{\lambda \in \mathbb{R}^n \,\middle|\, S_j(\lambda) > 0, \quad j = 1, \dots, k\right\} \tag{3}$$

and we denote by Γ_k^* the dual cone, given by

$$\Gamma_k^* = \left\{\lambda \in \mathbb{R}^n \,|\, \lambda \cdot \mu \geq 0, \quad \forall \mu \in \Gamma_k\right\} \tag{4}$$

Corresponding dual functions ρ_k, ρ_k^* are defined by

$$\rho_k(\lambda) = \left\{\frac{S_k(\lambda)}{\binom{n}{k}}\right\}^{1/k} \tag{5}$$

$$\rho_k^*(\lambda) = \inf\left\{\frac{\lambda \cdot \mu}{n} \,\middle|\, \mu \in \Gamma_k,\ \rho_k(\mu) \geq 1\right\}.$$

For matrices $\mathcal{A} \in \mathcal{S}^n$, we write $\mathcal{A} \in \Gamma_k\,(\Gamma_k^*)$ if the eigenvalues of $\mathcal{A}$, $\lambda = \lambda(\mathcal{A}) \in \Gamma_k(\Gamma_k^*)$, $\rho_k(\mathcal{A}) = \rho_k(\lambda)$ and $\rho_k^*(\mathcal{A}) = \rho_k^*(\lambda)$. Our main theorem may then be formulated as follows.

Theorem 1 *Let $u \in C^{2,1}(\mathcal{D}) \cap C^0(\overline{\mathcal{D}})$ satisfy the differential inequality*

$$Pu \leq f \tag{6}$$

in $\mathcal{D}$ for some coefficient matrix $\mathcal{A} \in \Gamma_k^*$, $n/2 < k \leq n$, at each point of $\mathcal{D}$, with $\rho_k^*(\mathcal{A}) > 0$ in $\mathcal{D}$. Suppose also $u \leq 0$ on $\partial_p\mathcal{D}$, the parabolic boundary of $\mathcal{D}$. Then we have the estimate

$$\sup_{\mathcal{D}} u \leq C \left\| \frac{f}{\rho^*(\mathcal{A})^{k/(k+1)}} \right\|_{L^{k+1}(\mathcal{D})}, \tag{7}$$

where C is a constant depending on n, k and $\mathcal{D}$.

Note that the parabolic boundary $\partial_p\mathcal{D}$ is defined by

$$\partial_p\mathcal{D} = \left\{(x,t) \in \partial\mathcal{D} \,\middle|\, (x,t') \notin \mathcal{D}, \quad \text{for any } t' < t\right\}. \tag{8}$$

When $k = n$, there is no restriction on $\mathcal{A}$ and we obtain the Krylov maximum principle [3].

As with the elliptic case [8], we will actually derive a sharp form of the estimate (7), using the Hessian measure theory of Trudinger and Wang [14], (see also [15,16]). Moreover the domain $\mathcal{D}$ in the right hand side of (7) may be replaced by an appropriate upper contact set as in the elliptic case. We postpone the cases, $k \leq n/2$, to a later treatment as there we cannot use the same method.

2 Parabolic Hessian equations

For $k = 1, \dots, n$, we define the operator P_k on $C^{2,1}(\mathcal{D})$ by

$$P_k[u] = -(D_t u)\, S_k(D^2 u). \tag{9}$$

Suppose now that $u \in C^2(\mathcal{D})$ satisfies the differential inequality (7). From Proposition 2.1 in [8], we have

$$\begin{aligned} P_k[-u] &= (D_t u)\, S_k(-D^2 u) \\ &\le \binom{n}{k} \frac{1}{\left(n\rho_k^*(\mathcal{A})\right)^k}\, D_t u \left(-a^{ij} D_{ij} u\right)^k \\ &\le \frac{k}{k+1} \binom{n}{k} \left\{ \left(n\rho_k^*(\mathcal{A})\right)^{\frac{-k}{k+1}} f \right\}^{k+1} \end{aligned} \tag{10}$$

provided $-D^2 u \in \overline{\Gamma_k},\ D_t u \ge 0,\ \mathcal{A} \in \Gamma_k^*$. Consequently, replacing u by $-u$, we have the differential inequality,

$$P_k[u] \le \psi, \tag{11}$$

where

$$\psi = \frac{k}{k+1} \binom{n}{k} \left[\frac{1}{n\rho_k^*(\mathcal{A})} \right]^k f^{k+1},$$

holding on the subset of $\mathcal{D}$ where $D^2 u \in \overline{\Gamma_k}$ and $D_t u \le 0$, that is where the function u is spatially k-convex and non-increasing in time. As with the elliptic case [8], the estimate is thus reduced to existence and estimation of solution of Hessian equations with inhomogeneous terms in L^p spaces. For the parabolic Hessian equation the relevant existence theorem is due to Reye [12], (with a thorough treatment given in the monograph [9]).

For our purpose here it is enough to formulate it in cylinders $Q_T = \Omega \times (0, T]$, where Ω is a smooth domain in $\mathbb{R}^n$, which is assumed to be uniformly $(k-1)$-convex, that is the principal curvatures $(\kappa_1, \ldots, \kappa_{n-1})$ of the boundary $\partial\Omega$, lie in the cone Γ_{k-1} in $\mathbb{R}^{n-1}$. The function $u \in C^{2,1}(\mathcal{D})$ is admissible with respect to the operator (9), or simply k-admissible, if $D_t u \le 0$ in $\mathcal{D}$ and $D^2 u(x,t) \in \overline{\Gamma_k}$ for each $(x,t) \in \mathcal{D}$.

Theorem 2 *Let $\psi_0 \in C^{2,1}(\overline{Q_T})$, $\inf \psi_0 > 0$ and Ω be uniformly $(k-1)$-convex with $\partial\Omega \in C^4$. Then there exists a unique k-admissible solution $u \in C^{2,1}(\overline{Q_T})$ of the initial boundary value problem,*

$$\begin{aligned} P_k[u] &= \psi_0, \quad \text{in} \quad Q_T \\ u &= 0 \quad \text{on} \quad \partial_p Q_T. \end{aligned} \tag{12}$$

Theorem 2 extends to parabolic equations of the form (2.4), the corresponding elliptic result due to Caffarelli et al. [1], used in [8]. To apply Theorem 2 to the proof of Theorem 1.1, we let $Q_0 = \Omega_0 \times [0, T]$ be a cylinder containing $\mathcal{D}$ with Ω_0 uniformly $(k-1)$-convex, with boundary $\partial\Omega_0 \in C^4$, and set

$$\psi' = P_k[u]\chi_{\mathcal{D}_\mathrm{k}}, \tag{13}$$

where $\mathcal{D}_\mathrm{k} = \mathcal{D}_\mathrm{k}^-$ is the lower k-contact set of u in $\mathcal{D}$ given by

$$\begin{aligned} \mathcal{D}_\mathrm{k}^- = \Big\{(x,t) \Big|\ &\exists\, k\text{−admissible } \ v \in C^{2,1}(\mathcal{D}) \ \text{ satisfying} \\ &v \le u \ \text{ in } \mathcal{D}, \ \ v(x,t) = u(x,t)\Big\}. \end{aligned} \tag{14}$$

Clearly for any $(x,t) \in \mathcal{D},\ D^2 u(x,t) \in \overline{\Gamma}_k$ and $D_t u(x,t) \le 0$. For $\psi_0 \in C^{2,1}(\overline{Q_0})$, satisfying $\psi_0 > 0$, in $\overline{Q_0},\ \ \psi' < \psi_0$ in $\mathcal{D}$, we define $u_0 \in C^{2,1}(\overline{Q_0})$ to be the unique k-admissible solution of (12). Analogously, to the elliptic case, we have $u \ge u_0$ in $\mathcal{D}$. To see this, we suppose there exists a point $(x_0, t_0) \in \overline{\mathcal{D}} - \partial_p\mathcal{D}$ such that

$$u_0(x_0, t_0) - u(x_0, t_0) = \sup_{\mathcal{D}} (u_0 - u) > 0.$$

Since u_0 is k-admissible, we must have $(x_0, t_0) \in \mathcal{D}_k^-$. But then $D^2u_0(x_0, t_0) \leq D^2u(x_0, t_0)$, $D_t u_0(x_0, t_0) \leq D_t u(x_0, t_0)$ implies $P_k[u_0] \leq P_k[u]$ at (x_0, t_0) which contradicts (13). Consequently, Theorem 1 will follow from corresponding estimates for k-admissible solutions of (12) with $\psi_0 \in L^1$, which we establish in the next section.

3 Estimates

In the previous section, we reduced the proof of Theorem 1, to L^∞ estimates for parabolic Hessian operators of the form (9). As with the case $k = n$, due to Krylov [3], we accomplish this through the Hessian integrals $I_k[u]$, introduced in [14,18]. For Ω a domain in $\mathbb{R}^n$ and $u \in C^2(\Omega)$, we define the Hessian integral $I_k[u]$, $k = 1, \ldots, n$, by

$$I_k[u] = -\int_\Omega u S_k(D^2u). \tag{15}$$

By integration by parts, we have for $u = 0$ on $\partial\Omega$,

$$I_k[u] = \int_\Omega S_k^{ij} D_i u D_j u, \tag{16}$$

where, for any matrix $r \in \mathcal{S}^n$,

$$S_k^{ij} = \frac{\partial}{\partial r_{ij}} S_k(r).$$

Accordingly, if u is k-convex in Ω, that is $D^2u \in \overline{\Gamma_k}$, then $[S_k^{ij}] \geq 0$ and hence $I_k[u] \geq 0$. To connect the Hessian integral with the operator (9), we let $u \in C^{2,1}(Q_T)$ and calculate

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} I_k[u(\cdot, t)] &= -\int_\Omega u_t S_k(D^2u) - \int_\Omega u S_k^{ij} D_{ijt} u \\ &= \int_\Omega P_k[u] - \int_\Omega u_t S_k^{ij} D_{ij} u \\ &= (k+1) \int_\Omega P_k[u] \end{aligned} \tag{17}$$

provided $u(\cdot, t) = 0$ on $\partial\Omega$. Here as for (19), we are using the key divergence free property

$$D_i(S_k^{ij}(D^2u)) = 0, \quad j = 1, \ldots, n. \tag{18}$$

Consequently, we have the identity,

$$I_k[u(\cdot, t)] = (k+1) \int_0^t \int_\Omega P_k[u] \tag{19}$$

for any $u \in C^{2,1}(Q_T)$ vanishing continuously on ∂Q_T. To proceed further, we need the monotonicity formula for Hessian integrals proved in [14], namely if $u \leq v$ in Ω, $u = v = 0$ on $\partial\Omega$ and $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ are k-convex in Ω, then

$$I_k[v] \leq I_k[u]. \tag{20}$$

From the weak continuity of Hessian integrals, also proved in [14], the monotonicity formula (20) extends to general k-convex functions $u, v \in C^0(\overline{\Omega})$, where

$$I_k[u] = -\int_\Omega u \, \mathrm{d}\mu_k[u] \tag{21}$$

and $\mu_k[u]$ denotes the k-Hessian measure of u. As in [8], we will make a specific choice for v in terms of the Green's function for the operator, $F_k[u] = S_k(D^2u)$. Indeed if $\partial\Omega$ is uniformly $(k-1)$-convex, $k > n/2$, there exists a unique k-convex function $G_y \in C^{0,2-n/k}(\overline{\Omega})$, for any point $y \in \Omega$, such that

$$\begin{aligned} \mu_k[G_y] &= \delta_y, \\ G_y &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{22}$$

where δ_y denotes the Dirac delta measure at y. Combining (19) and (20) with the specific choice

$$v = \left[\frac{u(y)}{G_y(y)}\right] G_y \geq u\,, \tag{23}$$

we thus obtain, for k-admissible $u \in C^{2,1}(Q_T)$,

$$\begin{aligned} [-u(y,t)]^{k+1} &\leq \left[-G_y(y)\right]^k I_k[u] \\ &\leq (k+1)\left|G_y(y)\right|^k \int_0^t \int_\Omega P_k[u]. \end{aligned} \tag{24}$$

Applying the estimate (24) to (12), we get the a priori estimate

$$-u(y,t) \leq \left\{(k+1)\left|G_y(y)\right|^k\right\}^{\frac{1}{k+1}} \left\{\int_{Q_T} \psi_0\right\}^{\frac{1}{k+1}} \tag{25}$$

for all $(y,t) \in Q_T$, and by letting ψ_0 approximate ψ', we obtain (7), by virtue of (10).

Moreover, defining the upper contact set $\mathcal{D}_k^+ = \mathcal{D}_k^-(-u)$ and taking account of (13), we may write (7) more precisely as

$$\begin{aligned} \sup_{\mathcal{D}} u &\leq \frac{1}{n}\left[nk\binom{n}{k}\right]^{\frac{1}{n+1}} \sup_{y\in\Omega}\left|G_y(y)\right|^{\frac{k}{k+1}} \left\|\frac{f}{\rho^*(\mathcal{A})^{k/(k+1)}}\right\|_{L^k(\mathcal{D}_k^+)} \\ &\leq \left\{\frac{k(\text{diam }\Omega)^{2k-n}}{[n(n-n/k)]^k\omega_n}\right\}^{\frac{1}{k+1}} \left\|\frac{f}{\rho^*(\mathcal{A})^{k/(k+1)}}\right\|_{L^k(\mathcal{D}_k^+)}, \end{aligned} \tag{26}$$

from formula (17) in [8]. The estimates (21), (26) constitute parabolic analogues of the precise estimates (21) and (22) in [8]. Because of the presence of the upper contact set, the estimate (26) also extends refinements of Krylov's estimate for $k = n$, obtained in [10–12,17]. As in [8], we may also express the dependence on the Green's function G in (26) in terms of the k-Hessian measure of the solution of the homogeneous equation in the punctured domain. We also note that discrete versions of the case $k = n$ are given in [5–7].

Finally we remark that from Theorem 1 we may derive corresponding extensions of local estimates for linear parabolic equations due to Krylov and Safonov [4], (see also [11–13]), as done for the elliptic case in [8].

References

1. Caffarelli, L., Nirenberg, L., Spruck, J.: The Dirichlet problem for nonlinear second order elliptic equations III: functions of the eigenvalues of the Hessian. Acta Math. **155**, 261–301 (1985)
2. Gilbarg, D., Trudinger, N.S.: Elliptic partial differential equations of second order, 2nd edn. Springer Verlag (1983)

3. Krylov, N.V.: Sequence of convex functions and estimates of maximum of the solution of a parabolic equation, Sibirsk. Mat. Z. **17**, 290–303 (1976) (Russian); English transl. in Siberian Math. J. **17**, 226–236 (1976)
4. Krylov, N.Y., Safonov, M.V.: Certain properties of solutions of parabolic equations with measurable coefficients. Izv. Akad. Nauk SSSR **44**, 161–175 (1980) (Russian); English transl. in Math. USSR-Izv. **16**, 151–164 (1981)
5. Kuo, H.J., Trudinger, N.S.: On the discrete maximum principle for parabolic difference operators. RAIRO Modél. Math. Anal. Numér. **27**, 719–737 (1993)
6. Kuo, H.J., Trudinger, N.S.: Evolving monotone difference operators on general space-time meshes. Duke Math. J. **91**, p587–607 (1998)
7. Kuo, H.J., Trudinger, N.S.: A note on the discrete Aleksandrov–Bakelman maximum principle. Taiwanese J. Math. **4**, 55–67 (2000)
8. Kuo, H.J., Trudinger, N.S.: New maximum principles for linear elliptic equations. Indiana Univ. Math. J. (to appear) http://www.iumj.indiana.edu/Preprints/3073.pdf
9. Lieberman, G.: Second order parabolic differential equations. World Scientific Publishing Co (1996) ISBN 981-02-2883-X
10. Nazarov, A.I., Ural′tseva, N.N.: Convex-monotone hulls and estimates of the maximum of the solution of parabolic equations. Zap. Nauchn Sem LOMI **147**, 95–109 (1985) (Russian); English tranl. in J. Soviet Math. **37**, 851–859 (1987)
11. Reye, S.J.: Harnack inequalities for parabolic equations in general form with bounded measurable coefficient. Research Report R44–84. Centre for Mathematical Analysis Australian National University (1984)
12. Reye, S.J.: Fully non-linear parabolic differential equations of second order. Doctor dissertation, Australian National University (1985)
13. Trudinger, N.S.: Local estimates for subsolutions and supersolutions of general second order elliptic quasilinear equations. Invent. Math. **61**, 67–79 (1980)
14. Trudinger, N.S., Wang, X.-J.: Hessian measures I. Topol. Methods Nonlin. Anal. **10**, 225–239 (1997)
15. Trudinger, N.S., Wang, X.-J.: Hessian measures II, Ann. Math. **150**, 579–604 (1999)
16. Trudinger, N.S., Wang, X.-J.: Hessian measures III. J. Funct. Anal. **193**, 1–23 (2002)
17. Tso, K.: On an Aleksandrov-Bakel'man type maximum principle for second order parabolic equations. Comm. Partial Differ. Eqs. **10**, 543–553 (1985)
18. Wang, X.-J.: A class of fully nonlinear elliptic equations and related functionals. Indiana Univ. Math. J. **43**, 25–54 (1994)